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GAPS IN THE DIMENSIONS OF ISOMETRY GROUPS OF RIEMANNIAN MANIFOLDS

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1. Introduction

If M is an n-dimensional Riemannian manifold and G is a closed subgroup of $I(M)$, the group of isometries of M, it is a classical result that

$$\dim G \leq \tfrac{1}{2}n(n+1) .$$

H. C. Wang [8] has shown that for $n \neq 4$, the dimension of G cannot be in the range:

$$\tfrac{1}{2}(n-1)n + 1 < \dim G < \tfrac{1}{2}n(n+1) ,$$

and H. Wakakuwa [9] has shown that for n large, the dimension of G cannot be in the range:

$$\tfrac{1}{2}(n-2)(n-1) + 3 < \dim G < \tfrac{1}{2}(n-1)n .$$

In this paper we generalize the results of Wang and Wakakuwa by showing

Theorem. *Let M be an n-dimensional Riemannian manifold with $n \neq 4$, 6, 10. Then the group $I(M)$ of isometries contains no closed subgroup G where the dimension of G falls into any of the ranges:*

$$\tfrac{1}{2}(n-k)(n-k+1) + \tfrac{1}{2}k(k+1) < \dim G < \tfrac{1}{2}(n-k+1)(n-k+2) , \quad k = 1, 2, 3, \cdots$$

The basic tool in the proof is our Theorem 2 of [4], which actually immediately implies the result for the special case where G is compact [2, p. 55].

2. The main results

We follow the terminology and notation of [3]. Let M be an n-dimensional Riemannian manifold and G a closed connected subgroup of $I(M)$, the group of isometries of M. For each $x \in M$ we let G_x denote the *isotropy subgroup of G at x*, and $G(x)$ the *G-orbit* of x. Then

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$$\dim G = \dim G_x + \dim G(x) .$$

If $G(x)$ is a G-orbit of highest dimension, it is known [3, Lemma 2.1] that G acts *essentially effectively* on $G(x)$. In other words, if K is the kernel of the action of G on $G(x)$, $\dim G/K = \dim G$ and G/K acts effectively on $G(x)$. This implies that

$$\dim G \leq \tfrac{1}{2}t(t+1) ,$$

where

$$t = \text{maximal dimension of the orbits of } G \text{ on } M .$$

We use the notation

$$\langle m \rangle = \tfrac{1}{2}m(m+1)$$

for m a positive integer. Let

$$\Phi(m) = \text{largest integer } j \text{ such that } \langle m-j \rangle + \langle j \rangle \leq \langle m-j+1 \rangle - 2 ,$$

$$\Psi(m) = \text{largest integer } j \text{ such that}$$

$$\langle m-j \rangle + \langle j \rangle \leq \langle m-j+1 \rangle + (j-1) - 2 .$$

(The symbol $\Phi(m)$ was introduced in [5].) It is easy to verify that for $m \geq 3$

$$\Phi(m) = [\tfrac{1}{2}(\sqrt{8m+1} - 3)] , \qquad \Psi(m) = [\tfrac{1}{2}(\sqrt{8m-15} - 1)] ,$$

$$\Psi(m) = \Phi(m-2) + 1 , \qquad \Psi(m) \geq \Phi(m) .$$

A short table of values of Ψ will be helpful later:

m	$\Psi(m)$
3	1
5	2
8	3

m	$\Psi(m)$
12	4
17	5
23	6

Theorem 1. *Let M be an n-dimensional Riemannian manifold, and G a closed connected subgroup of $I(M)$ acting on M with orbits of maximal dimension $n-l$, $0 \leq l \leq \Psi(n) - 1$. If* $\dim G$ *falls into any of the following ranges:*

$$\langle n-k \rangle + \langle k \rangle - l < \dim G < \langle n-k+1 \rangle + (k-1) - l ,$$
$$k = l+1, l+2, \cdots, \Psi(n) ,$$

where $\Psi(n)$ is the largest value of k for which the above inequalities are meaningful, then we must have $n \leq 12$ and exactly one of the possibilities below:

(1) $n=12$, $l=0$ (*i.e.*, *G acts transitively on M*), $\dim G=47$, $G_x^0=SU(6)$. [Example. $M=R^{12}$, $G=SU(6)\cdot R^{12}$ where the dot represents the semi-direct product.]

(2) $n=10$, $l=0$, $\dim G=35$, $G_x^0=U(5)$.
[Examples. $M=P_5(C)$, $G=SU(6)$; $M=R^{10}$, $G=U(5)\cdot R^{10}$.]

(3) $n=8$, $l=0$, $\dim G=22$, $G_x^0=G_2$.
[Examples. $M=S^7\times S^1$, $G=\text{Spin}(7)\times S^1$; $M=R^8$, $G=G_2\cdot R^8$.]

(4) $n=8$, $l=1$, $\dim G=21$, $G_x^0=G_2$.
[Examples. $M=S^7\times S^1$, $G=\text{Spin}(7)$; $M=R^8$, $G=G_2\cdot R^7$.]

(5) $n=7$, $l=0$, $\dim G=21$, $G_x^0=G_2$.
[Examples. $M=S^7$, $G=\text{Spin}(7)$; $M=R^7$, $G=G_2\cdot R^7$.]

(6) $n=6$, $l=0$, $\dim G=15$, $G_x^0=U(3)$.
[Examples. $M=P_3(C)$, $G=SU(4)$; $M=R^6$, $G=U(3)\cdot R^6$.]

(7) $n=6$, $l=0$, $\dim G=14$, $G_x^0=SU(3)$.
[Examples. $M=S^6$ or $P_6(R)$, $G=G_2$; $M=R^6$, $G=SU(3)\cdot R^6$.]

(8) $n=4$, $l=0$, $\dim G=8$, $G_x^0=U(2)$.
[Examples. $M=P_2(C)$, $G=SU(3)$; $M=R^4$, $G=U(2)\cdot R^4$.]

Proof. Let $x\in M$ such that $\dim G(x)=n-l$, and suppose $\dim G$ is in the range

$$\text{(a)}\qquad \langle n-k\rangle+\langle k\rangle-l<\dim G<\langle n-k+1\rangle+(k-1)-l$$

for some fixed $k, l+1\le k\le \Psi(n)$. Now

$$\text{(b)}\qquad \dim G_x^0=\dim G-(n-l)\ ,$$

and the compact connected Lie group G_x^0 acts effectively on M with a fixed point x. Therefore the maximal dimension t_1 of the orbits of G_x^0 on M is at most $n-1$.

Case A: $t_1=n-1$. It follows that $l=0$ and G acts transitively on M. From (a) we have

$$\text{(c)}\qquad \langle n-k\rangle+\langle k\rangle<\dim G<\langle n-k+1\rangle+(k-1)\ .$$

Now G_x^0 leaves invariant $(n-1)$-spheres in a neighborhood of the fixed point x. Therefore the principal orbit of the action of G_x^0 on M must be an $(n-1)$-sphere, and G_x^0 is now determined since the compact connected Lie groups which act transitively and effectively on topological spheres have been completely classified [6], [1], [7]. We have the following cases to consider:

(i) $G_x^0=SO(n)$, $n\ge 2$,
(ii) $G_x^0=SU(\frac{1}{2}n)$ or $U(\frac{1}{2}n)$, n even and $n\ge 4$,
(iii) $G_x^0=Sp(\frac{1}{4}n)$, $Sp(\frac{1}{4}n)\times S^1$ or $Sp(\frac{1}{4}n)\times Sp(1)$, n divisible by 4, $n\ge 4$,
(iv) $G_x^0=G_2$, $n=7$,
(v) $G_x^0=\text{Spin}(7)$, $n=8$,

(vi) $G_x^0 = \mathrm{Spin}\,(9)$, $n = 16$.

We consider these cases individually:

(i) We have

$$\dim G = \dim G_x^0 + n = \dim SO(n) + n = \langle n \rangle .$$

Hence $\dim G$ is not in the range (*c*).

(ii) For n even and $n \geq 14$,

$$\dim G \leq \dim U(\tfrac{1}{2}n) + n = \tfrac{1}{4}n^2 + n \leq \langle n - \Psi(n) \rangle + \langle \Psi(n) \rangle ,$$

so we need only consider the cases $n \leq 12$. Investigation turns up possibilities (1), (2), (6), (7) and (8) of the theorem.

(iii) For n divisible by 4 and $n \geq 8$,

$$\begin{aligned} \dim G &\leq \dim Sp(\tfrac{1}{4}n) + \dim Sp(1) + n = \tfrac{1}{8}n^2 + \tfrac{5}{4}n + 3 \\ &\leq \langle n - \Psi(n) \rangle + \langle \Psi(n) \rangle , \end{aligned}$$

so we need only consider $n = 4$. We obtain possibility (8) again.

(iv) Possibility (5) arises here.

(v) Here

$$\dim G = \dim \mathrm{Spin}\,(7) + 8 = 29 ,$$

and 29 does not fall into the range (*c*) for $n = 8$, $1 \leq k \leq 3$.

(vi) Here

$$\dim G = \dim \mathrm{Spin}\,(9) + 16 = 52, \text{ but } 52 < \langle 16 - \Psi(16) \rangle .$$

Case B: $t_1 \leq n - 2$. Let M_0 be a principal orbit of the action of G_x^0 on M, and let

$$\dim M_0 = t_1 = n - 2 - u , \qquad u \geq 0 .$$

If $u > 0$, we replace M_0 by

$$M_1 = M_0 \times S^u ,$$

so in any case we may assume G_x^0 acts effectively on a manifold of dimension exactly $n - 2$.

From (a) and (b), after simplification we obtain

(d) $$\langle n_1 - k_1 \rangle + \langle k_1 \rangle < \dim G_x^0 < \langle n_1 - k_1 + 1 \rangle ,$$

where $n_1 = n - 2$, $k_1 = k - 1$.

Observe

$$k_1 \leq \Psi(n) - 1 = \Phi(n_1) .$$

We may apply [4, Theorem 2] to the action of G_x^0 on M_1 to arrive at the following possibilities:

(i) $n_1 = 4$, $G_x^0 = SU(3)/Z$, $M_1 = P_2(C)$,

(ii) $n_1 = 6$, $G_x^0 = G_2$, $M_1 = S^6$ or $P_6(R)$,

(iiii) $n_1 = 10$, $G_x^0 = SU(6)/Z$, $M_1 = P_5(C)$.

(Z denotes the centers of $SU(3)$ and $SU(6)$.) Since $P_2(C)$, S^6, $P_6(R)$ and $P_5(C)$ do not split, we have

$$M_1 = M_0 .$$

But G_x^0 acts linearly in a neighborhood of its fixed point x and therefore cannot have $P_2(C)$, $P_6(R)$ or $P_5(C)$ as a principal orbit. We are left with possibilities (3) and (4) of the theorem.

Theorem 2. *Let M be an n-dimensional Riemannian manifold, and G a closed subgroup of $I(M)$. If* $\dim G$ *falls into any of the following ranges:*

$$\langle n-k\rangle + \langle k\rangle < \dim G < \langle n-k+1\rangle , \qquad k = 1, 2, \cdots, \Phi(n) ,$$

(note that $\Phi(n)$ is the largest integer k for which the above inequalities are meaningful), then we must have $n = 4, 6$ or 10, G acting transitively on M and exactly one of the possibilities below:

(1) $n = 10$, $k = 3$, $\dim G = 35$, $G_x^0 = U(5)$,

(2) $n = 6$, $k = 2$, $\dim G = 14$, $G_x^0 = SU(3)$,

(3) $n = 4$, $k = 1$, $\dim G = 8$, $G_x^0 = U(2)$.

Proof. Suppose

$$\langle n-k_0\rangle + \langle k_0\rangle < \dim G < \langle n-k_0+1\rangle$$

for some k_0, $1 \leq k_0 \leq \Phi(n)$. Let

$$n - l_0 = \text{maximum dimension of the orbits of } G \text{ on } M .$$

Since

$$\langle n-l_0\rangle \geq \dim G > \langle n-k_0\rangle \geq \langle n-\Phi(n)\rangle ,$$

we have

$$l_0 < k_0 \leq \Phi(n) \leq \Psi(n) .$$

Clearly

$$\langle n-k_0\rangle + \langle k_0\rangle - l_0 < \dim G < \langle n-k_0+1\rangle + (k_0-1) - l_0 .$$

But we are now precisely in the situation of Theorem 1. It follows that $n \leq 12$

and that we have one of the eight possibilities of Theorem 1. It is easily checked that only possibilities (2), (7) and (8) of Theorem 1 survive.

Remark. Using [3, § 7] it is possible to obtain much sharper characterizations of the exceptional low-dimensional cases in Theorems 1 and 2. If G is compact, the exceptional cases of Theorem 2 are given precisely by [4, Theorem 2].

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